3. Find a term in an infinite series of rational parallelopipeds where the edges are in proportion as 2 : 3 : 9, within unity in *length*.

Let $2x$, $3x$, and $9x\pm1$ be the edges. $94x^2\pm18x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp18)/(94-m^2)$. Substitute $m=1464/151$, and $x=15855$, $2x=31710$, $3x=47565$, $9x-1=142694$.

Proof : $31710^2+47565^2+142694^2=153719^2$.

4. Find some term in an infinite series of rational parallelopipeds where the dimensions come within 1 unit in the *thickness* of being in proportion as 3 : 6 : 7.

Let edges be $3x\pm1$, $6x$ and $7x$. $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp6)/(94-m^2)$.

When $m=29/3$	$m=126/33$	
$x=24$	$x=429$	
$3x\pm1=144$	$3x\pm1=1286$	etc.
$6x=144$	$6x=2574$	
$7x=168$	$7x=3003$	
S. d. $=233$	S. d. $=4159$	

Proof : $73^2+144^2+168^2=233^2$.

5. Find some term in an infinite series of rational rectangular solids where the edges come within 1 unit in the *width* of being in the proportion of 3 : 6 : 7. Let the edges be represented by $3x$, $6x\pm1$ and $7x$. Then $94x^2\pm12x+1=\square=(mx\pm1)^2=m^2x^2+2mx+1$. $x=(2m\mp12)/(94-m^2)$. When $m=\sqrt{94}$ ......1464/151. Then $x=84258$ or 357870.

$3x=252774$	or $3x=1073610$
$6x-1=505547$	$6x+1=2147221$
$7x=589806$	$7x=2505090$
Diagonal $=816911$	Diagonal $=3469679$

6. Find a term in that infinite series of rational parallelopipeds wherein the edges of every solid are within unity in the *length* of being in proportion to each other as 3 : 6 : 7.

$(3x)^2+(6x)^2+(7x\pm1)^2=94x^2\pm14x+1=\square=(mx\pm1)^2$.

$94x\pm14=m^2x\pm2m$. $x=(2m\mp14)/(94-m^2)$. $m=\sqrt{94}$. Now when $m=29/3$, $x=60$, $3x=180$, $6x=360$, $7x-1=419$.

$180^2+360^2+419^2=581^2$.

Also solved by *J. H. DRUMMOND*.

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers, [i. e. triangular numbers that are also square numbers], equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate. Or, if s and t be the roots of any two successive triangular number that are also square numbers, prove that $t-s=2n+1$, where $n^2(n+1)^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$$\frac{n(n+1)}{2} \text{ is a square when } n=\frac{(1+\sqrt{2})^{2m}+(1-\sqrt{2})^{2m}-2}{4}.$$

$$\therefore \pm\sqrt{\frac{n(n+1)}{2}}=\pm\left\{\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

$$\pm\sqrt{\frac{n'(n'+1)}{2}}=\pm\left\{\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

Taking (2)+ and (1)−, and then taking their difference, we easily get,

$$\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}+\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}=2y+1.$$

$$\therefore \frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{2}=2y+1.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}-\tfrac{1}{2}\right\}^2+$$

$$\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}+\tfrac{1}{2}\right\}^2=y^2+(y+1)^2.$$

$$\therefore 2\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}\right\}^2+\tfrac{1}{2}=y^2+(y+1)^2.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}-(1-\sqrt{2})^{2m+1}}{2\sqrt{2}}\right\}^2=y^2+(1y+1)^2.$$

In above m can have any positive integral value.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

This problem is true if we read "The sum of" instead of "The difference between." It might also be stated as follows: The difference between the roots of two successive triangular square numbers equals a number whose square is the sum of the squares of two successive integral numbers.

From Solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is $n(n+1)/2$, the next in order, in terms of n, is $\left(2n+1+3\sqrt{\frac{n(n+1)}{2}}\right)^2$.

The *difference* of the two roots is $2n+1+2\sqrt{\frac{n(n+1)}{2}}$.

The *sum* of the two roots is $2n+1+4\sqrt{\frac{n(n+1)}{2}}$, which equals the sum of the two consecutive integral numbers, $n+2\sqrt{\frac{n(n+1)}{2}}$ and $n+1+2\sqrt{\frac{n(n+1)}{2}}$.

But $\left(n+2\sqrt{\frac{n(n+1)}{2}}\right)^2+\left(n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2=6n^2+6n+1+(8n+4)\sqrt{\frac{n(n+1)}{2}}$

which equals the square of the *difference* of the two roots, or

$$\left(2n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2$$

Illustration.—From the series of triangular square numbers, 1^2, 6^2, 35^2, 204^2, 1189^2, etc., take 6 and 35. $35-6=29$; $35+6=41=20+21$; $20^2+21^2=29^2$.

This problem and problems No. 45, (Vol. III., No. 5, page 153), and No. 36, of Diophantine Analysis, are very closely related.

Also solved by the *PROPOSER*.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbiam University, Washington, D. C.

Prove that a "magic square" of nine integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three.

I. Solution by M. W. HASKELL, M. A., Ph. D., Associate Professor of Mathematics, University of California, Berkeley, California.

Let the magic square be

a	b	c
d	e	f
g	h	k

and let S be the constant sum.

Then $S=a+b+c=d+e+f=g+h+k=a+d+g=b+e+h=c+f+k=a+e+k=c+e+g$.

Adding these all together, we have $8S=3a+2b+3c+2d+4e+2f+3g+2h+3k=3(a+c+g+k)+2(b+e+h)+2(d+e+f)$. But the last two quantities in parenthesis are each$=S$. Hence $4S=3(a+c+g+k)$, and S is a multiple of 3.

II. Solution by —— (Paper Unsigned.)

Suppose the numbers occupying the magic square to be a, b, c, d, e, f, g, h, k. Now $a+e+k=b+e+h=c+e+g=S$.

$\therefore$ $a+k\equiv k(\text{mod } 3)$, $b+h\equiv k(\text{mod } 3)$, $c+g\equiv k(\text{mod } 3)$, where $S-e\equiv k(\text{mod } 3)$.

Adding the congruences, $(a+b+c)+(g+h+k)\equiv 0(\text{mod } 3)$. Or, since $(a+b+c)+(g+h+k)\equiv 0(\text{mod } 3)$, $2S\equiv 0(\text{mod } 3)$.

Multiply by 2, and divide by 3, and the result is $S\equiv 0$. Q. E. D.

III. Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Let the rows of the "square" be a, b, c ; x, y, z ; and l, m, n, and let the constant sum be k. We have to show that $k/3$ is integral. We have $a+y+n=k$; $b+y+m=k$; $l+y+c=k$. Add, and we have $(a+b+c)+(l+m+n)+3y=3k$, that is, $2k+3y=3k$.

$\therefore$ $3y=k$. $\therefore$ $y=k/3$. But y is integral. $\therefore$ $k/3$ is integral.

Also solved by *M. A. GRUBER* and *G. B. M. ZERR*.